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Frames, the β-duality in Minkowski space and spin coherent states

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Abstract. In the spirit of some earlier work on building coherent states for the Poincaré group in one space and one time dimension, we construct here analogous families of states for the full Poincaré group, for representations corresponding to mass $m > 0$ and arbitrary integral or half-integral spin. Each family of coherent states is defined by an affine section in the group and constitutes a frame. The sections, in their turn, are determined by particular velocity vector fields, the latter always appearing in dual pairs. Geometrically, each family of coherent states is related to the choice of a Riemannian structure on the forward mass hyperboloid or, equivalently, to the choice of a certain parallel bundle in the relativistic phase space. The large variety of coherent states obtained tempts us to believe that there is rich scope here for application to spin-dependent problems in atomic and nuclear physics, as well as to image reconstruction problems, using the discretized versions of these frames.

1. Introduction

Following some earlier work [3, 4], *coherent states* (CS) will be associated in this paper with particular types of *square integrable* representations of groups (see also [6] for a general review of the theory). Specifically, let G be a locally compact group and $g \mapsto U(g)$ a unitary, irreducible representation (UIR) of G on the (complex, separable) Hilbert space $\mathfrak{H}$. Let $H \subset G$ be a closed subgroup, $X = G/H$ the left coset space, which will be assumed to carry the *invariant* measure ν (under the natural action of G on X). Suppose that there exists a (finite) set of vectors η^i, $i = 1, 2, \ldots, n$, in $\mathfrak{H}$ and a Borel section $\sigma : X \to G$, such that

$$\sum_{i=1}^{n} \int_X |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}| \, \mathrm{d}\nu(x) = A \qquad \eta^i_{\sigma(x)} = U(\sigma(x))\eta^i \tag{1.1}$$

where A is a bounded positive operator on $\mathfrak{H}$, with a densely defined inverse. The integral in (1.1) is assumed to converge weakly. We then say that the representation U is *square*

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integrable mod(H, σ), and call the set of vectors

$$\mathfrak{S} = \{\eta^i_{\sigma(x)} \mid x \in X,\ i = 1, 2, \ldots, n\} \subset \mathfrak{H} \tag{1.2}$$

a *family of covariant coherent states* (CS for short), for the representation U. If it so happens that A^{-1} is also a bounded operator, then we say that the family of CS, $\mathfrak{S}$, forms a *rank-n frame*, denoted $\mathcal{F}\{\eta^i_{\sigma(x)}, A, n\}$. If A is a multiple of the identity operator I on $\mathfrak{H}$, we say that the frame is *tight*. In this case, by appropriately normalizing the η^i we may actually make $A = I$, and then (1.1) is said to generate a *resolution of the identity*.

Covariant CS for the Poincaré group $\mathcal{P}^\uparrow_+(1,1)$, in one space and one time dimension, for UIRs corresponding to mass $m > 0$, have been studied exhaustively in [4, 5]. These states are labelled by points of the homogeneous space, $\Gamma = \mathcal{P}^\uparrow_+(1,1)/T$ (T = time translation subgroup), and each family of CS is associated with a certain *affine section* $\sigma : \Gamma \to \mathcal{P}^\uparrow_+(1,1)$. The set of all affine sections itself enjoys interesting mathematical properties, including a type of *duality*, which groups members of the set into pairs. In particular, there exists a *symmetric* section which is self-dual. The full Poincaré group $\mathcal{P}^\uparrow_+(1,3)$ had been studied earlier, and a special class of CS was obtained in [7, 13]. These CS were built out of UIRs corresponding to mass $m > 0$ and spin $s = 0, 1, 2, \ldots$. In the context of the $\mathcal{P}^\uparrow_+(1,1)$ CS, these other CS for $\mathcal{P}^\uparrow_+(1,3)$, were all related to one particular section, $\sigma = \sigma_0$, the Galilean section. Some early results on constructing CS for the spin-$\frac{1}{2}$, $m > 0$ representation of $\mathcal{P}^\uparrow_+(1,3)$ were reported in [14]. Because of the particular choice of the measure on the phase space Γ, used in that paper, the resulting CS did not constitute a frame, and hence could not generate a resolution of the identity. Nevertheless, the results obtained there show how bispinor (or Dirac) type CS may be constructed, allowing one, among other things, to arrive at a Dirac equation on the phase space. Coherent states for the De Sitter and Poincaré groups, arising from particular sections were also studied in [10].

The present paper extends the results in [4, 5] for the $\mathcal{P}^\uparrow_+(1,1)$ CS, to any UIR of $\mathcal{P}^\uparrow_+(1,3)$, for $m > 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Just as in the $\mathcal{P}^\uparrow_+(1,1)$ situation, each family of CS is associated with a particular affine section, and one always obtains a frame in this manner. The final result is enunciated in proposition 7.3. Moreover, for specific situations, the frame can be made tight, thus leading to a resolution of the identity. Since the base space, on which the sections are now defined, is much larger than in the $\mathcal{P}^\uparrow_+(1,1)$ case, the full significance of the duality mentioned above becomes clear. Each affine section is characterized by a 4-vector field $u(p) = (u_0(p), \boldsymbol{u}(p))$, ($p$ is a relativistic 4-momentum on the mass shell), which maps the forward mass hyperboloid $\mathcal{V}^+_m$ to itself. The *dual section*, is then characterized by the 4-vector field $u^*(p)$, obtained by applying the Lorentz boost Λ_p to $\overline{u}(p) = (u_0(p), -\boldsymbol{u}(p))$, pointwise for all p. Furthermore, each section is seen to define a *Riemannian structure* on the forward mass hyperboloid $\mathcal{V}^+_m$, considered as a manifold. This Riemannian structure is, in its turn, the restriction to the tangent bundle of $\mathcal{V}^+_m$ of a pseudo-Riemannian metric of signature $(3, 1)$ on an ambient vector bundle, which is also a homogeneous space of $\mathcal{P}^\uparrow_+(1,3)$. It turns out that we have an illustration here of the use of parallel bundles, in the sense of [9], for constructing CS.

The result is that obtain a wide class of families of CS for the Poincaré group, for arbitrary spin. The richness in their variety tempts us to believe that they would prove useful in studies of spin related effects in atomic or nuclear physics, as well as in more theoretical studies on quantization of systems with internal degrees of freedom. It ought to be pointed out that the CS obtained in this paper are similar to the vector CS studied in [16]. However, the situation envisaged here is more general than that in [16], since in our case, the subspace generated by the ‘fiducial vectors’ is not stable under the action of any non-trivial subgroup of $\mathcal{P}^\uparrow_+(1,3)$.

2. Notational conventions

By the full Poincaré group we mean the 2-fold covering group, $\mathcal{P}_+^\uparrow(1,3) = T^4 \oslash SL(2,\mathbb{C})$, where $T^4 \simeq \mathbb{R}_{1,3}$ is the group of space-time translations. Elements of $\mathcal{P}_+^\uparrow(1,3)$ will be denoted by (a, A), with $a = (a_0, \boldsymbol{a}) \in \mathbb{R}_{1,3}$, $\quad A \in SL(2,\mathbb{C})$. The multiplication law is $(a, A)(a', A') = (a + \Lambda a', AA')$, where $\Lambda \in \mathcal{L}_+^\uparrow(1,3)$ (the proper, orthochronous Lorentz group) is the Lorentz transformation corresponding to A:

$$\Lambda^\mu{}_\nu = \tfrac{1}{2}\,\mathrm{Tr}[A\sigma_\nu A^\dagger \sigma_\mu] \qquad \mu, \nu = 0, 1, 2, 3 \tag{2.1}$$

$\sigma^1 = \sigma_x,\ \sigma^2 = \sigma_y$ and $\sigma^3 = \sigma_z$ are the Pauli matrices, $\sigma^0 = \mathbb{I}_2$, and the metric tensor is $g_{00} = 1 = -g_{11} = -g_{22} = -g_{33}$. Let

$$\mathcal{V}_m^+ = \{k = (k_0, \boldsymbol{k}) \in \mathbb{R}_{1,3} \mid k^2 = k_0^2 - \boldsymbol{k}^2 = m^2\} \tag{2.2}$$

be the *forward mass hyperboloid* (we take $c = \hbar = 1$). Then

$$k' = \Lambda k \Rightarrow \sigma \cdot k' = A\sigma \cdot kA^\dagger \tag{2.3}$$

with $\sigma \cdot k = \sigma^\mu k_\mu = k_0\mathbb{I}_2 - \boldsymbol{k} \cdot \boldsymbol{\sigma},\ \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

In the Wigner realization, the unitary, irreducible representation U_W^s of $\mathcal{P}_+^\uparrow(1,3)$ for a particle of mass $m > 0$ and spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, is carried by the Hilbert space

$$\mathfrak{H}_W^s = \mathbb{C}^{2s+1} \otimes L^2\left(\mathcal{V}_m^+, \frac{\mathrm{d}\boldsymbol{k}}{k_0}\right) \tag{2.4}$$

of $\mathbb{C}^{2s+1}$-valued functions ϕ on $\mathcal{V}_m^+$, which are square integrable:

$$\int_{\mathcal{V}_m^+} \phi(k)^\dagger \phi(k)\, \frac{\mathrm{d}\boldsymbol{k}}{k_0} = \|\phi\|^2 = \langle \phi|\phi\rangle < \infty\,. \tag{2.5}$$

Explicitly

$$(U_W^s(a, A)\phi)(k) = \mathrm{e}^{\mathrm{i}k\cdot a}\, \mathcal{D}^s(h(k)^{-1}Ah(\Lambda^{-1}k))\phi(\Lambda^{-1}k) \qquad k \cdot a = k_0 a_0 - \boldsymbol{k} \cdot \boldsymbol{a} \tag{2.6}$$

where $\mathcal{D}^s$ is the $(2s + 1)$-dimensional irreducible spinor representation of $SU(2)$ (carried by $\mathbb{C}^{2s+1}$) and

$$k \to h(k) = \frac{m\mathbb{I}_2 + \sigma \cdot \bar{k}}{\sqrt{2m(k_0 + m)}} \qquad (\bar{k} = (k_0, -\boldsymbol{k})) \tag{2.7}$$

is the image in $SL(2,\mathbb{C})$ of the Lorentz boost Λ_k, which brings the 4-vector $(m, \boldsymbol{0})$ to the 4-vector k in $\mathcal{V}_m^+$:

$$\begin{aligned}\Lambda_k(m, \boldsymbol{0}) = k \Leftrightarrow h(k)\, m\mathbb{I}_2\, h(k) &= m[h(k)]^2 \\ &= \sigma \cdot \bar{k} = \begin{pmatrix} k_0 + k_z & k_x - \mathrm{i}k_y \\ k_x + \mathrm{i}k_y & k_0 - k_z \end{pmatrix}.\end{aligned} \tag{2.8}$$

The matrix form of the Lorentz boost is

$$\Lambda_k = \frac{1}{m}\begin{pmatrix} k_0 & \boldsymbol{k}^\dagger \\ \boldsymbol{k} & mV_k \end{pmatrix} = \Lambda_k{}^\dagger \tag{2.9}$$

where V_k is the 3×3 symmetric matrix

$$V_k = \mathbb{I}_3 + \frac{\boldsymbol{k} \otimes \boldsymbol{k}^\dagger}{m(k_0 + m)} = V_k{}^\dagger = V_{\bar{k}}\,. \tag{2.10}$$

The following properties of Λ_k and V_k are useful for computational purposes and are easily verified:

(i) $\det \Lambda_k = 1$; Λ_k has the spectrum

$$\mathrm{Sp}(\Lambda_k) = \left\{ \frac{1}{m}(k_0 \pm \|\boldsymbol{k}\|),\ 1,\ 1 \right\} \quad \Rightarrow \quad \|\Lambda_k\| = \frac{1}{m}(k_0 + \|\boldsymbol{k}\|)\,. \tag{2.11}$$

(ii) The matrix V_k has the properties

$$\det V_k = \frac{k_0}{m} \qquad \|V_k\| = \frac{k_0}{m} \qquad \|V_k^{-1}\| = 1 \tag{2.12}$$

$$V_k^{-1} = V_{\bar{k}}^{-1} = V_k - \frac{\boldsymbol{k} \otimes \boldsymbol{k}^\dagger}{mk_0}. \tag{2.13}$$

Furthermore

$$(\Lambda_k p)_0 = \frac{1}{m}(k_0 p_0 + \boldsymbol{k} \cdot \boldsymbol{p}) = \frac{k \cdot \bar{p}}{m} \qquad \underline{(\Lambda_k p)} = \frac{1}{m}\boldsymbol{k} p_0 + V_k \boldsymbol{p} \tag{2.14}$$

the underline denoting the spatial part of a 4-vector, while

$$k_0 \underline{(\Lambda_k p)} - \boldsymbol{k}(\Lambda_k p)_0 = k_0 V_k^{-1} \boldsymbol{p} \tag{2.15}$$

from which it follows that

$$\left\| k_0 \underline{(\Lambda_k p)} - \boldsymbol{k}(\Lambda_k p)_0 \right\| \leqslant k_0 \|\boldsymbol{p}\|. \tag{2.16}$$

Also, since $\Lambda_k^{-1} = \Lambda_{\bar{k}}$, we have, $h(k)^{-1} = h(\bar{k})$.

3. Coherent states for massive, spin-*s* particles on the phase space

A classical (spinless), relativistic particle has a phase space which can be identified with

$$\Gamma = \mathcal{P}_+^\uparrow(1,3)/(T \times SU(2)) \tag{3.1}$$

where T denotes the subgroup of time translations. For a particle with non-zero spin (treated as an additional classical degree of freedom), a geometric quantization programme [19] would normally start with the phase space $\Gamma' = \mathcal{P}_+^\uparrow(1,3)/T \times SO(2)$. However, since geometric quantization is not our objective here, we choose Γ in (3.1) as the phase space for a particle with arbitrary spin s. (In the terminology of geometric quantization, this means working on a $\mathbb{C}^{2s+1}$-bundle, rather than on a line bundle.) For $A \in SL(2,\mathbb{C})$, let

$$A = h(k)R(k) \qquad R(k) \in SU(2) \tag{3.2}$$

be its Cartan decomposition. An arbitrary element $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$ has the left coset decomposition

$$(a, A) = \left(\left(0, \boldsymbol{a} - \frac{a_0 \boldsymbol{k}}{k_0}\right),\ h(k) \right) \left(\left(\frac{m a_0}{k_0}, \boldsymbol{0} \right),\ R(k) \right) \tag{3.3}$$

according to (3.1). Thus, the elements in Γ have the global coordinatization, $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^6$:

$$\boldsymbol{q} = \boldsymbol{a} - \frac{a_0 \boldsymbol{k}}{k_0} \qquad \boldsymbol{p} = \boldsymbol{k}. \tag{3.4}$$

In terms of these variables, the action of $\mathcal{P}_+^\uparrow(1,3)$ on Γ is given by $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q}', \boldsymbol{p}') = (a, A)(\boldsymbol{q}, \boldsymbol{p})$

$$\boldsymbol{q}' = \frac{1}{p_0'}\left(p_0'[\boldsymbol{a} + \underline{\Lambda(0, \boldsymbol{q})}] - \boldsymbol{p}'[a_0 + \{\Lambda(0, \boldsymbol{q})\}_0]\right)$$
$$\boldsymbol{p}' = \underline{\Lambda p} \qquad p = (\sqrt{m^2 + \boldsymbol{p}^2},\ \boldsymbol{p}) \tag{3.5}$$

where $\Lambda \in \mathcal{L}_+^\uparrow(1,3)$ is related to A by (2.1), and $p_0' = (\Lambda p)_0$. It can be shown [2] that the measure $\mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}$ is invariant under this action, and hence represents the invariant measure ν on Γ, in the variables $(\boldsymbol{q}, \boldsymbol{p})$.

Next, in terms of these variables let us define the basic section

$$\sigma_0 : \Gamma \to \mathcal{P}_+^\uparrow(1,3) \qquad \text{with} \quad \sigma_0(\boldsymbol{q}, \boldsymbol{p}) = ((0, \boldsymbol{q}),\ h(p)) \tag{3.6}$$

which we call the *Galilean section*. Any other section $\sigma : \Gamma \to \mathcal{P}_+^\uparrow(1,3)$ is then related to σ_0 in the manner

$$\sigma(\boldsymbol{q}, \boldsymbol{p}) = \sigma_0(\boldsymbol{q}, \boldsymbol{p})\,((f(\boldsymbol{q}, \boldsymbol{p}),\ \boldsymbol{0}),\ R(\boldsymbol{q}, \boldsymbol{p})) \tag{3.7}$$

where $f : \mathbb{R}^6 \to \mathbb{R}$ and $R : \mathbb{R}^6 \to SU(2)$ are smooth functions. As in the case of $\mathcal{P}_+^\uparrow(1,1)$, we work again with *affine sections*, for which the function f is of the form

$$f(\boldsymbol{q}, \boldsymbol{p}) = \varphi(\boldsymbol{p}) + \boldsymbol{q} \cdot \boldsymbol{\vartheta}(\boldsymbol{p}) \tag{3.8}$$

where $\varphi : \mathbb{R}^3 \to \mathbb{R}$, $\boldsymbol{\vartheta} : \mathbb{R}^3 \to \mathbb{R}^3$ are smooth functions of $\boldsymbol{p}$ alone. Again, it can be shown that as far as the construction of CS is concerned, φ only introduces an inessential phase. Hence, we set $\varphi = 0$. Moreover, we also impose the restriction that $R(\boldsymbol{q}, \boldsymbol{p}) = R(\boldsymbol{p})$ be a function of $\boldsymbol{p}$ alone. Writing

$$\sigma(\boldsymbol{q}, \boldsymbol{p}) = (\hat{q},\ h(p)R(\boldsymbol{p})) \qquad \hat{q} = (\hat{q}_0,\ \hat{\boldsymbol{q}}) \in \mathbb{R}_{1,3} \tag{3.9}$$

we see that

$$\hat{q}_0 = \boldsymbol{\beta}(\boldsymbol{p}) \cdot \hat{\boldsymbol{q}} \tag{3.10}$$

where $\boldsymbol{\beta}$ is the 3-vector field

$$\boldsymbol{\beta}(\boldsymbol{p}) = \frac{p_0 \boldsymbol{\vartheta}(\boldsymbol{p})}{m + \boldsymbol{p} \cdot \boldsymbol{\vartheta}(\boldsymbol{p})} \qquad \text{so that} \qquad \boldsymbol{\vartheta}(\boldsymbol{p}) = \frac{m\boldsymbol{\beta}(\boldsymbol{p})}{p_0 - \boldsymbol{p} \cdot \boldsymbol{\beta}(\boldsymbol{p})}. \tag{3.11}$$

Let us also introduce the *dual* vector field $\boldsymbol{\beta}^*$

$$\boldsymbol{\beta}^*(\boldsymbol{p}) = \frac{\boldsymbol{p} - mV_p\boldsymbol{\beta}(\boldsymbol{p})}{p_0 - \boldsymbol{p} \cdot \boldsymbol{\beta}(\boldsymbol{p})}, \tag{3.12}$$

where V_p is the matrix defined in (2.10). The significance of this dual quantity will shortly become clear. First note that

$$\boldsymbol{\beta}^{**} = \boldsymbol{\beta} \qquad \text{and} \qquad \boldsymbol{\vartheta}(\boldsymbol{p}) = \frac{1}{m}[\boldsymbol{p} - mV_p\boldsymbol{\beta}^*(\boldsymbol{p})]. \tag{3.13}$$

We now take an arbitrary affine section σ, and going back to the Hilbert space $\mathfrak{H}_W^s$ in (2.4) choose a set of vectors $\boldsymbol{\eta}^i$, $i = 1, 2, \ldots, 2s+1$, in it to define the formal operator (see equations (1.1) and (2.6)):

$$A_\sigma = \sum_{i=1}^{2s+1} \int_{\mathbb{R}^6} |\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})}\rangle\langle\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})}|\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p} \qquad \boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})} = U_W^s(\sigma(\boldsymbol{q}, \boldsymbol{p}))\boldsymbol{\eta}^i. \tag{3.14}$$

From the general definition, in order for the set of vectors

$$\mathfrak{S}_\sigma = \{\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})} \mid (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^6,\ i = 1, 2, \ldots, 2s+1\} \subset \mathfrak{H}_W^s \tag{3.15}$$

to constitute a family of coherent states for the representation U_W^s, the integral in (3.14) must converge weakly, and define A_σ as a bounded operator with inverse. In fact, it will be possible to choose vectors $\boldsymbol{\eta}^i$ such that for each affine section σ, both A_σ and A_σ^{-1} are bounded, i.e. each family of CS, $\mathfrak{S}_\sigma$, will define a rank-$(2s+1)$ frame.

To study the convergence properties of the operator integral in (3.14) we have to determine the convergence of the ordinary integral

$$I_{\phi,\psi} = \sum_{i=1}^{2s+1} \int_{\mathbb{R}^6} \langle \boldsymbol{\phi} | \boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})} \rangle \langle \boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})} | \boldsymbol{\psi} \rangle \, \mathrm{d}\boldsymbol{q} \, \mathrm{d}\boldsymbol{p} \tag{3.16}$$

for arbitrary $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathfrak{H}_W^s$. In (3.9) set

$$\hat{A}(p) = h(p) R(\boldsymbol{p}) \qquad \text{and} \qquad \hat{\Lambda}(p) = \Lambda_p \rho(p) \tag{3.17}$$

where $\hat{\Lambda}(p)$ and $\rho(p)$ are the matrices in the Lorentz group $\mathcal{L}_+^\uparrow(1,3)$ which correspond to $\hat{A}(p)$ and $R(\boldsymbol{p})$, respectively. Then

$$\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})}(k) = \exp\{-\mathrm{i}\boldsymbol{X}(\boldsymbol{k}) \cdot \boldsymbol{q}\} \mathcal{D}^s(v(k,p)) \boldsymbol{\eta}^i (\hat{\Lambda}(p)^{-1} k) \tag{3.18}$$

where

$$\boldsymbol{X}(\boldsymbol{k}) = \boldsymbol{k} - \frac{k \cdot p}{m} \boldsymbol{\vartheta}(\boldsymbol{p}) \tag{3.19}$$

and

$$v(k,p) = h(k)^{-1} \hat{A}(p) h(\hat{\Lambda}(p)^{-1} k) \in SU(2). \tag{3.20}$$

Substituting into (3.16) yields

$$I_{\phi,\psi} = \sum_{i=1}^{2s+1} \int_{\mathbb{R}^6 \times \mathcal{V}_m^+ \times \mathcal{V}_m^+} \exp\{-\mathrm{i}[\boldsymbol{X}(\boldsymbol{k}) - \boldsymbol{X}(\boldsymbol{k}')] \cdot \boldsymbol{q}\} \, \boldsymbol{\phi}(k)^\dagger \mathcal{D}^s(v(k,p))$$

$$\times \boldsymbol{\eta}^i (\hat{\Lambda}(p)^{-1} k) \boldsymbol{\eta}^i (\hat{\Lambda}(p)^{-1} k')^\dagger \mathcal{D}^s(v(k',p))^\dagger \boldsymbol{\psi}(k') \, \frac{\mathrm{d}\boldsymbol{k}}{k_0} \, \frac{\mathrm{d}\boldsymbol{k}'}{k_0'} \, \mathrm{d}\boldsymbol{q} \, \mathrm{d}\boldsymbol{p}. \tag{3.21}$$

In order to perform the k, k' integrations in (3.21), we need to change variables: $\boldsymbol{k} \to \boldsymbol{X}(\boldsymbol{k})$. Computing the Jacobian $\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})$ of this transformation from (3.19) we obtain

$$\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k}) = \left(\frac{\partial X_i}{\partial k_j} (\boldsymbol{k}) \right) = \mathbb{I}_3 + \frac{1}{m k_0} \boldsymbol{\vartheta}(\boldsymbol{p}) \otimes [k_0 \boldsymbol{p} - \boldsymbol{k} p_0]^\dagger \tag{3.22}$$

which has the determinant

$$\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] = 1 + \frac{1}{m k_0} \boldsymbol{\vartheta}(\boldsymbol{p}) \cdot [k_0 \boldsymbol{p} - \boldsymbol{k} p_0]. \tag{3.23}$$

Since at $\boldsymbol{k} = \boldsymbol{p} = \boldsymbol{0}$, $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] = 1$, and since in order to change variables we need $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] \neq 0$, we must impose the condition that

$$\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] > 0 \qquad \forall \ (\boldsymbol{k}, \boldsymbol{p}). \tag{3.24}$$

This, in turn, imposes restrictions on $\boldsymbol{\vartheta}$, and hence on the 4-vector $\hat{q} = (\hat{q}_0, \hat{\boldsymbol{q}})$ in (3.9), which we proceed to study next.

4. The β-duality and space-like sections

Rewriting (3.23) in terms of $\boldsymbol{\beta}^*$ (see the appendix)

$$\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] = \frac{p_0(\Lambda_k p)_0}{mk_0}\left[1 + \frac{(\Lambda_k \overline{p})}{(\Lambda_k p)_0}\cdot\rho(k\to\overline{p})^\dagger\boldsymbol{\beta}^*(\boldsymbol{p})\right] \tag{4.1}$$

where $\rho(k\to\overline{p})$ is the rotation matrix

$$\rho(k\to\overline{p}) = \Lambda_p^{-1}\Lambda_k\Lambda_p\Lambda_k^{-1}. \tag{4.2}$$

Thus, the positivity of $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})]$, as required by (3.24), would be ensured if the second term within the square brackets in (4.1) did not exceed 1 in magnitude, i.e. if $\|\boldsymbol{\beta}^*(\boldsymbol{p})\| < 1,\ \forall\,\boldsymbol{p}$. This is made precise in the following proposition (we give the proof in the appendix).

Proposition 4.1. The following conditions are equivalent:
(i) The 4-vector $\hat{q} = (\hat{q}_0,\ \hat{\boldsymbol{q}})$ is space-like, i.e.

$$|\hat{q}_0|^2 - \|\hat{\boldsymbol{q}}\|^2 < 0. \tag{4.3}$$

(ii) The matrix

$$S(\boldsymbol{p},\boldsymbol{\vartheta}) = \mathbb{I}_3 + \left[\boldsymbol{\vartheta}\otimes\left(\frac{\boldsymbol{p}}{m} - \frac{\boldsymbol{\vartheta}}{2}\right)^\dagger + \left(\frac{\boldsymbol{p}}{m} - \frac{\boldsymbol{\vartheta}}{2}\right)\otimes\boldsymbol{\vartheta}^\dagger\right] \tag{4.4}$$

is positive definite for all $p \in \mathcal{V}_m^+$.
(iii) For all $\boldsymbol{p} \in \mathbb{R}^3$, the 3-vector field $\boldsymbol{\beta}$ obeys

$$\|\boldsymbol{\beta}(\boldsymbol{p})\| < 1. \tag{4.5}$$

(iv) For all $\boldsymbol{p} \in \mathbb{R}^3$, the 3-vector field $\boldsymbol{\beta}^*$ obeys

$$\|\boldsymbol{\beta}^*(\boldsymbol{p})\| < 1. \tag{4.6}$$

Consequently, we have the next result (again, we give the proof in the appendix).

Proposition 4.2. The condition (3.24), $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] > 0$, holds for all $\boldsymbol{k}, \boldsymbol{p} \in \mathbb{R}^3$ if and only if the 4-vector $\hat{q} = (\hat{q}_0,\ \hat{\boldsymbol{q}})$ is space-like, i.e. if and only if any one of the equivalent conditions in proposition 4.1 is satisfied.

An affine section σ, for which the corresponding 3-vector $\boldsymbol{\vartheta}$ or $\boldsymbol{\beta}$ satisfies any one of the equivalent conditions of proposition 4.1 will be called a *space-like affine section*. We could now go back to (3.21) and, assuming proposition 4.1 to hold, carry out the change of variables to get estimates on $|I_{\phi,\psi}|$, thereby arriving at conditions under which (3.14) would define a bounded invertible operator A_σ. This would ensure that (3.15) does indeed define a family of CS for the representation U_W^s of $\mathcal{P}_+^\uparrow(1,3)$. However, let us first analyse, in some detail, the geometry of the sections and the relativistic meaning of the duality between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^*$.

5. Geometry of the β-duality; Riemannian structures

The group $\mathcal{P}_+^\uparrow(1,3)$ has the action $x \mapsto x' = (a, A)x = a + \Lambda x$, on T^4. Considering T^4 as a manifold, we identify its tangent space $T_x(T^4)$, at any point x, with T^4 itself. The derivative $\boldsymbol{T}_x(a, A)$ of the map $x \mapsto (a, A)x$ at $x \in T^4$ is simply Λ. Let T^{4*} be the dual space of T^4 and consider $T^4 \times T^{4*}$ as the cotangent bundle of T^4. (Note: While in conformity with standard usage, we use an asterisk to denote the dual space, this does not reflect the duality implied by the $\boldsymbol{\beta}^*$ of the last section). Then any $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$ induces the action $(x, x^*) \mapsto (a + \Lambda x, \Lambda^* x^*)$ on $(x, x^*) \in T^4 \times T^{4*}$, where Λ^* is defined by $\langle x^*; \Lambda x\rangle = \langle \Lambda^* x^*; x\rangle$, $(\forall\ x \in T^4)$, $\langle \cdot\, ; \,\cdot\rangle$ being the dual pairing between T^4 and T^{4*}. In particular, take $(x, x^*) = (0, 1_m)$, 1_m being the (non-zero) element in T^{4*} left invariant by rotations, i.e. $\rho^* 1_m = 1_m$, where $\rho \in \mathcal{L}_+^\uparrow(1,3)$ is a rotation. The orbit of this point, in $T^4 \times T^{4*}$, under the action of $\mathcal{P}_+^\uparrow(1,3)$, can be identified with $T^4 \times \mathcal{V}_m^+$. Note also that, since the tangent space $T_{x^*}(T^{4*})$, to T^{4*} at a point $x^* \in T^{4*}$, can again be identified with T^{4*} itself, and since $\mathcal{V}_m^+ \subset T^{4*}$, it follows that at any $p \in \mathcal{V}_m^+$, the tangent space $T_p(\mathcal{V}_m^+) \subset T^{4*}$ and the cotangent space $T_p^*(\mathcal{V}_m^+) \subset T^4$. Consider $T^4 \times \mathcal{V}_m^+$ as a vector bundle with base space $\mathcal{V}_m^+$ and fibre T^4 above $p \in \mathcal{V}_m^+$. The sub-bundle

$$W = \{(x,\, p) \in T^4 \times \mathcal{V}_m^+ \mid \langle w^*; x\rangle = 0,\ \forall\ w^* \in T_p(\mathcal{V}_m^+)\} \tag{5.1}$$

is called a *normal bundle* over $\mathcal{V}_m^+$ [1, 9], and we denote the fibre of W, above $p \in \mathcal{V}_m^+$, by W_p. (W_p is simply the annihilator of the tangent space $T_p(\mathcal{V}_m^+)$ at p.) A sub-bundle Σ of $T^4 \times \mathcal{V}_m^+$ is called a *parallel bundle* if

$$W \oplus \Sigma = T^4 \times \mathcal{V}_m^+ \tag{5.2}$$

the sum being understood in the sense that $W_p \oplus \Sigma_p = T^4$, as vector spaces, ($\Sigma_p$ is the fibre of Σ at p).

In terms of the coordinates p_i, $i = 1, 2, 3$, of $p = (p_0, \boldsymbol{p}) \in \mathcal{V}_m^+$, the tangent space $T_p(\mathcal{V}_m^+)$ is spanned by the three vectors

$$\frac{\partial}{\partial p_i} = \frac{1}{m}\left(\frac{p_i}{p_0}, \hat{\boldsymbol{e}}_i\right) \tag{5.3}$$

where $\hat{\boldsymbol{e}}_1 = (1, 0, 0)$, $\hat{\boldsymbol{e}}_2 = (0, 1, 0)$ and $\hat{\boldsymbol{e}}_3 = (0, 0, 1)$. We add to this set the vector

$$\frac{\partial}{\partial p_0} = \frac{1}{m}\,(1, \mathbf{0}) \tag{5.4}$$

to make $\{\partial/\partial p_\mu\}_{\mu=0}^3$ into a basis for T^{4*}. Similarly, the cotangent space $T_p^*(\mathcal{V}_m^+)$ is spanned by the covectors

$$\mathrm{d}p_i = m\,(0, \hat{\boldsymbol{e}}_i) \tag{5.5}$$

to which we add the covector

$$\mathrm{d}p_0 = m\left(1, -\frac{\boldsymbol{p}}{p_0}\right) \tag{5.6}$$

and obtain the dual basis $\{\mathrm{d}p_\mu\}_{\mu=0}^3$ for T^4:

$$\left\langle \mathrm{d}p_\mu\,;\ \frac{\partial}{\partial p_\nu}\right\rangle = \delta_{\mu\nu}. \tag{5.7}$$

Thus, the general tangent vectors $X_p \in T_p(\mathcal{V}_m^+)$ and covectors $X_p^* \in T_p^*(\mathcal{V}_m^+)$ are of the form

$$X_p = m\sum_{i=1}^3 q_i\,\frac{\partial}{\partial p_i} = \left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{p_0},\ \boldsymbol{q}\right) \in T^{4*} \qquad X_p^* = m\sum_{i=1}^3 q_i\ \mathrm{d}p_i = m^2(0, \boldsymbol{q}) \in T^4. \tag{5.8}$$

(Here q_i is assumed to have the dimension of a length). When T^{4*} is equipped with the Minkowski metric g (with diagonal elements $1, -1, -1, -1$), $T_p(\mathcal{V}_m^+)$ becomes a space-like hyperplane:

$$X_p \cdot X_p = X_p^\dagger g X_p = \frac{|\boldsymbol{q}\cdot\boldsymbol{p}|^2}{p_0^2} - \|\boldsymbol{q}\|^2 < 0 \tag{5.9}$$

with (Minkowski) normal vector $\hat{n} = (1/m)(p_0, \boldsymbol{p})$:

$$\hat{n} \cdot X_p = 0 = \langle \mathrm{d}p_0\,; X_p \rangle. \tag{5.10}$$

In other words, the pseudo-metric $-g$, of signature $(3, 1)$, (with three positive and one negative eigenvalues) on T^{4*} restricts to a positive definite metric on $T_p(\mathcal{V}_m^+)$, for each $p \in \mathcal{V}_m^+$. Furthermore, as a consequence of (5.10), the normal bundle W is seen to be

$$W = \{(\lambda\, \mathrm{d}p_0, p) \in T^4 \times \mathcal{V}_m^+ \mid \lambda \in \mathbb{R}\} \qquad W_p = \left\{\lambda\, \mathrm{d}p_0 = m\lambda \left(1, -\frac{\boldsymbol{p}}{p_0}\right) = \lambda \overline{p} \mid \lambda \in \mathbb{R}\right\}. \tag{5.11}$$

Consequently, a possible choice for the parallel bundle Σ is

$$\Sigma = \{(X_p^*, p) \in T^4 \times \mathcal{V}_m^+ \mid X_p^* \in T_p^*(\mathcal{V}_m^+)\} = T^*(\mathcal{V}_m^+) \qquad \Sigma_p = T_p^*(\mathcal{V}_m^+). \tag{5.12}$$

However, this is not the only possible choice for a parallel bundle, as we now demonstrate.

Since $\boldsymbol{\beta}(\boldsymbol{p})$ in (3.11) satisfies (4.5), let us define the relativistic 4-velocity

$$n(p) = (n_0(\boldsymbol{p}), \boldsymbol{n}(\boldsymbol{p})) \qquad n_0(\boldsymbol{p}) = [1 - \|\boldsymbol{\beta}(\boldsymbol{p})\|^2]^{-\frac{1}{2}} \qquad \frac{\boldsymbol{n}(\boldsymbol{p})}{n_0(\boldsymbol{p})} = \boldsymbol{\beta}(\boldsymbol{p}). \tag{5.13}$$

Then, by (3.10) the point $\hat{q} = (\hat{q}_0, \hat{\boldsymbol{q}}) \in T^4$ satisfies

$$n(p) \cdot \hat{q} = 0 \tag{5.14}$$

i.e. $\hat{q}$ lies on the space-like hyperplane with normal vector $n(p)$, determined by $\boldsymbol{\beta}(\boldsymbol{p})$. Let Σ_p^β denote this hyperplane. In particular, for the *Galilean section*

$$\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_0(\boldsymbol{p}) = \boldsymbol{0} \qquad \sigma = \sigma_0 \qquad \Sigma_p^\beta = \Sigma_p^0 = \{(0, \boldsymbol{q}) \mid \boldsymbol{q} \in \mathbb{R}^3\} = T_p^*(\mathcal{V}_m^+) \tag{5.15}$$

while for the *Lorentz section*

$$\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_\ell(\boldsymbol{p}) = \frac{\boldsymbol{p}}{p_0} \qquad \sigma = \sigma_\ell \qquad \Sigma_p^\beta = \Sigma_p^\ell = \left\{\hat{q} \in T^4 \,\middle|\, \frac{1}{m}\, p \cdot \hat{q} = 0\right\} \tag{5.16}$$

so that Σ_p^ℓ is in fact isomorphic to the tangent space $T_p(\mathcal{V}_m^+)$ (see (5.10). Note that these two sections are related by the duality of (3.12), i.e. $\boldsymbol{\beta}_0^*(\boldsymbol{p}) = \boldsymbol{\beta}_\ell(\boldsymbol{p})$.

There is a natural isomorphism $D_p : T^4 \to T^{4*}$, mapping $T_p^*(\mathcal{V}_m^+)$ to $T_p(\mathcal{V}_m^+)$ with

$$D_p\, \mathrm{d}p_\mu = \frac{\partial}{\partial p_\mu} \qquad \mu = 0, 1, 2, 3. \tag{5.17}$$

Similarly, there is an isomorphism $F_p^\beta : T^4 \to T^4$, mapping $\Sigma_p^0 = T_p^*(\mathcal{V}_m^+)$ to Σ_p^β with

$$F_p^\beta(0, \boldsymbol{q}) = \hat{q} \qquad F_p^\beta(1, \boldsymbol{0}) = (1, \boldsymbol{\beta}(\boldsymbol{p})) \tag{5.18}$$

with $\hat{q}$ coming from (3.9):

$$F_p^\beta = \begin{pmatrix} 1 & \dfrac{p_0}{m}\, \boldsymbol{\vartheta}(\boldsymbol{p})^\dagger \\[2ex] \boldsymbol{\beta}(\boldsymbol{p}) & \mathbb{I}_3 + \dfrac{\boldsymbol{p} \otimes \boldsymbol{\vartheta}(\boldsymbol{p})^\dagger}{m} \end{pmatrix}. \tag{5.19}$$

Since $\overline{p} = (p_0, -\boldsymbol{p})$ is a time-like vector, W_p is a one-dimensional time-like subspace of T^4, and since Σ_p^β is space-like, equation (5.18) implies that $W_p \oplus \Sigma_p^\beta = T^4$. Hence

$$\Sigma^\beta = \{(X_p^{*\beta}, p) \in T^4 \times \mathcal{V}_m^+ \mid X_p^{*\beta} \in \Sigma_p^\beta\} \tag{5.20}$$

is a parallel bundle, which we call a *space-like parallel bundle*. In other words, each $\boldsymbol{\beta}$ determines a *bundle isomorphism* $(X_p^*, p) \mapsto (F_p^\beta X_p^*, p) = (X_p^{*\beta}, p)$ between the parallel bundles $\Sigma = T_p^*(\mathcal{V}_m^+)$ and Σ^β. The corresponding affine section σ, (see equation (3.9)), maps the phase space Γ in (3.1) to $\sigma(\Gamma)$, the latter being identifiable with the parallel bundle Σ^β. In particular, when $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, Γ is identified with $T_p^*(\mathcal{V}_m^+)$, while when $\boldsymbol{\beta} = \boldsymbol{\beta}_\ell$, it is identified with the parallel bundle Σ^ℓ which is isomorphic to $T_p(\mathcal{V}_m^+)$.

To sum up, each space-like affine section σ determines a space-like parallel bundle, identifiable with its range, and the coherent states $\boldsymbol{\eta}_{\sigma(q,p)}$ (see equation (3.15)) may be thought of as being labelled by the points of this bundle. Each space-like parallel bundle is determined by a map $u : \mathcal{V}_m^+ \to \mathcal{V}_m^+$, $u(p) = m\{n(p)\}$, with $\boldsymbol{u}(\boldsymbol{p})/u_0(\boldsymbol{p}) = \boldsymbol{\beta}(\boldsymbol{p})$ and $n(p)$ given by (5.13). The two extreme cases of this map arise when $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, giving rise to the Galilean section, and its dual $\boldsymbol{\beta} = \boldsymbol{\beta}_\ell$, yielding the Lorentz section. In the first case, u is the constant map, $u(p) = (m, \mathbf{0})$, $\forall\ p \in \mathcal{V}_m^+$, while in the second, u is the identity map, $u(p) = p$, $\forall\ p \in \mathcal{V}_m^+$. Physically, when $\sigma = \sigma_0$, the phase space Γ may be thought of as consisting of all positions and momenta $(\boldsymbol{q}, \boldsymbol{p})$, of the particle of mass m, when it is observed from a fixed laboratory frame. In the case where $\sigma = \sigma_\ell$, Γ is to be identified with the set of all space-time and momentum coordinates $(\hat{q}, \boldsymbol{p})$ of the particle, where $\hat{q}$ is observed from a frame which is moving with the velocity, $-\boldsymbol{p}/p_0$, i.e. opposite to the particle. A general section σ identifies the phase space with the set of all space-time and momentum coordinates $(\hat{q}, \boldsymbol{p})$, with $\hat{q}$ being observed from a frame moving, with respect to the laboratory, with velocity $-\boldsymbol{\beta}(\boldsymbol{p})$.

The map F_p^β in (5.19) equips the vector bundle $T^4 \times \mathcal{V}_m^+$ with a pseudo-metric, the restriction of which to $T^*(\mathcal{V}_m^+) \subset T^4 \times \mathcal{V}_m^+$ yields a positive definite metric on the fibres $T_p^*(\mathcal{V}_m^+)$. Indeed, using equations (5.4), (5.8) and (5.18) we may rewrite (5.14) as

$$\left\langle F_p^{\beta\dagger}(-g) F_p^\beta X_p^* ;\ \frac{\partial}{\partial p_0} \right\rangle = 0 = (1, \mathbf{0})\ [F_p^{\beta\dagger}(-g)F_p^\beta] \begin{pmatrix} 0 \\ \boldsymbol{q} \end{pmatrix}. \tag{5.21}$$

Hence, $T^*(\mathcal{V}_m^+)$ is the hyperplane in T^4 with normal vector $(1, \mathbf{0})$, with respect to the pseudo-metric $F_p^{\beta\dagger}(-g)F_p^\beta$. Let

$$\widetilde{R}^\beta = \sum_{\mu,\nu=0}^{3} (\widetilde{R}_p^\beta)_{\mu\nu} \frac{\partial}{\partial p_\mu} \otimes \frac{\partial}{\partial p_\nu} \tag{5.22}$$

where $(\widetilde{R}_p^\beta)_{\mu\nu}$ are the matrix elements of the operator

$$\widetilde{R}_p^\beta = F_p^{\beta\dagger}(-g)F_p^\beta$$

$$= \begin{pmatrix} -1 + \|\boldsymbol{\beta}(\boldsymbol{p})\|^2 & -\dfrac{p_0}{m}\boldsymbol{\vartheta}(\boldsymbol{p})^\dagger + \boldsymbol{\beta}(\boldsymbol{p})^\dagger \left[\mathbb{I}_3 + \dfrac{\boldsymbol{\vartheta}(\boldsymbol{p}) \otimes \boldsymbol{p}^\dagger}{m}\right] \\ -\dfrac{p_0}{m}\boldsymbol{\vartheta}(\boldsymbol{p}) + \left[\mathbb{I}_3 + \dfrac{\boldsymbol{p} \otimes \boldsymbol{\vartheta}(\boldsymbol{p})^\dagger}{m}\right]\boldsymbol{\beta}(\boldsymbol{p}) & S(\boldsymbol{p}, \boldsymbol{\vartheta}) \end{pmatrix}$$

expressed in the $\{\partial/\partial p_\mu\}$–$\{\mathrm{d}p_\mu\}$ bases ($S(\boldsymbol{p}, \boldsymbol{\vartheta})$ being as in (4.4))

$$(\widetilde{R}_p^\beta)_{\mu\nu} = \mathrm{d}p_\mu^\dagger \widetilde{R}_p^\beta \mathrm{d}p_\nu. \tag{5.23}$$

In view of (5.18), when restricted to $\Sigma_p^0 = T_p^*(\mathcal{V}_m^+)$, $\widetilde{R}_p^\beta$ reduces to $S(\boldsymbol{p}, \boldsymbol{\vartheta})$, and thus $\widetilde{R}_p^\beta$ defines a positive definite metric on the cotangent bundle $T_p^*(\mathcal{V}_m^+)$. Next write $(R_p^\beta)^{\mu\nu} = (\widetilde{R}_p^\beta)_{\mu\nu}$. Then

$$R^\beta = \sum_{\mu,\nu=0}^{3} (R_p^\beta)^{\mu\nu} \mathrm{d}p_\mu \otimes \mathrm{d}p_\nu \tag{5.24}$$

defines a pseudo-metric on $T^{4*} \times \mathcal{V}_m^+$. Also, it is clear that R^β restricts to a positive definite metric on $T(\mathcal{V}_m^+)$ and thereby equips $\mathcal{V}_m^+$ with a *Riemannian structure*. In fact, the restriction of R^β to $T_p(\mathcal{V}_m^+)$ is again $S(\boldsymbol{p}, \boldsymbol{\vartheta})$. Comparing with (5.9), we see that the affine section σ determined by the 3-vector field $\boldsymbol{\beta}$ has the effect of changing the pseudo-metric $-g$ on T^{4*} to R^β, both having signature $(3, 1)$ and both being positive definite on the tangent space $T_p(\mathcal{V}_m^+)$.

6. Kinematic interpretation of the β-duality

Let us try to get a more physical understanding of the $\boldsymbol{\beta} - \boldsymbol{\beta}^*$ duality in (3.11), (3.12). From equations (5.13) and (5.14), we see that, with $\|\boldsymbol{\beta}(\boldsymbol{p})\| < 1$, $(\forall\ \boldsymbol{p})$, we can associate with $\boldsymbol{\beta}(\boldsymbol{p})$ the time-like 4-vector field $u(p) = m\{n(p)\}$ which is normal to the space-like hyperplane Σ_p^β in T^4. More generally, if $\boldsymbol{p} \mapsto \boldsymbol{\beta}(\boldsymbol{p})$ is a $\boldsymbol{p}$-dependent 3-vector field, we can associate with it a ray $[u(p)]$ of p-dependent relativistic 4-vector fields $u(p)$ in the following way:

$$\begin{aligned} &u(p) = (u_0(\boldsymbol{p}), \boldsymbol{u}(\boldsymbol{p})) \qquad p = (p_0, \boldsymbol{p}) \\ &u_0(\boldsymbol{p}) > 0 \qquad \frac{\boldsymbol{u}(\boldsymbol{p})}{u_0(\boldsymbol{p})} = \boldsymbol{\beta}(\boldsymbol{p}) \end{aligned} \tag{6.1}$$

with $u(p) \cdot u(p) = u_0(\boldsymbol{p})^2 - \|\boldsymbol{u}(\boldsymbol{p})\|^2$. The 4-vector $u(p)$, and hence the ray $[u(p)] = \mathbb{R}^+\{u(p)\}$ is time-like, light-like or space-like according as $\|\boldsymbol{\beta}(\boldsymbol{p})\| < 1$, $\|\boldsymbol{\beta}(\boldsymbol{p})\| = 1$ or $\|\boldsymbol{\beta}(\boldsymbol{p})\| > 1$, respectively. Under a Lorentz transformation Λ, $u(p) \mapsto \Lambda u(p)$ and

$$\boldsymbol{\beta}(\boldsymbol{p}) = \frac{\boldsymbol{u}(\boldsymbol{p})}{u_0(\boldsymbol{p})} \mapsto \boldsymbol{\beta}'(\boldsymbol{p}) = \frac{\overline{\Lambda u(p)}}{(\Lambda u(p))_0}. \tag{6.2}$$

Of course, such a transformation preserves the property of u being time-like, light-like or space-like, and hence of the equivalent properties of $\|\boldsymbol{\beta}(\boldsymbol{p})\|$ being $<$, $=$ or > 1. For any $u(p) \in [u(p)]$, the 4-vector field

$$u^*(p) = \Lambda_p \overline{u(p)} \tag{6.3}$$

depends on p only, (like $u(p)$ itself). We call $u^*(p)$ the *dual* of the 4-vector field $u(p)$. Then

$$u(p) = \Lambda_p \overline{u^*(p)} \qquad u^{**}(p) = u(p) \qquad \boldsymbol{\beta}^*(\boldsymbol{p}) = \frac{\boldsymbol{u}^*(\boldsymbol{p})}{u_0^*(\boldsymbol{p})} \tag{6.4}$$

and $\boldsymbol{\beta}(\boldsymbol{p})$, $\boldsymbol{\beta}^*(\boldsymbol{p})$ satisfy the duality relationship in (3.12). Taking the dot product of $\boldsymbol{\beta}(\boldsymbol{p})$ with $\boldsymbol{p}$ on both sides of (3.12), using the explicit form for V_p in (2.10), and rearranging, we get the relation

$$(p_0 - \boldsymbol{p} \cdot \boldsymbol{\beta}(\boldsymbol{p}))\,(p_0 - \boldsymbol{p} \cdot \boldsymbol{\beta}^*(\boldsymbol{p})) = m^2 \tag{6.5}$$

which has an interesting physical interpretation, as we shall see in (6.20). Similarly, one may verify the matrix relation

$$(mV_p - \boldsymbol{p} \otimes \boldsymbol{\beta}(\boldsymbol{p})^\dagger)(mV_p - \boldsymbol{p} \otimes \boldsymbol{\beta}^*(\boldsymbol{p})^\dagger) = m^2 \mathbb{I}_3 \tag{6.6}$$

which has a complementary physical interpretation (see (6.19). Note also, that $u^*(p)$ is time-like if $u(p)$ is time-like and vice-versa. Hence

$$\|\boldsymbol{\beta}^*(\boldsymbol{p})\| < 1 \quad \Leftrightarrow \quad \|\boldsymbol{\beta}(\boldsymbol{p})\| < 1\,. \tag{6.7}$$

Physically, with each ordinary 3-vector velocity $\boldsymbol{\beta}(\boldsymbol{p})$, $u(p)$ associates a relativistic 4-velocity $n(p)$ $(= u(p)/[u(p) \cdot u(p)]^{\frac{1}{2}}$, if $u(p) \cdot u(p) \neq 0$ and $= u(p)/u_0(p)$ if $u(p) \cdot u(p) =$

0), while $\boldsymbol{\beta}^*(\boldsymbol{p})$ is the velocity obtained by relativistically adding the 3-velocity $-\boldsymbol{\beta}(\boldsymbol{p})$ to the 3-velocity associated with the boost Λ_p.

Below are details of some particular space-like affine sections and a light-like limiting section, all of which have interesting physical interpretations.

(1) The Galilean section σ_0

As noted in equation (5.15), for this section

$$\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_0(\boldsymbol{p}) = \mathbf{0} \quad \boldsymbol{\vartheta}(\boldsymbol{p}) = \boldsymbol{\vartheta}_0(\boldsymbol{p}) = \mathbf{0} \qquad \boldsymbol{\beta}_0^*(\boldsymbol{p}) = \frac{\boldsymbol{p}}{p_0} \quad \boldsymbol{\vartheta}_0^*(\boldsymbol{p}) = \frac{\boldsymbol{p}}{m}. \tag{6.8}$$

Here, $\|\boldsymbol{\beta}_0(\boldsymbol{p})\| < 1,\ \|\boldsymbol{\beta}_0^*(\boldsymbol{p})\| < 1,\ \forall\ \boldsymbol{p}$.

(2) The Lorentz section σ_ℓ

This time (see equation (5.16))

$$\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_\ell(\boldsymbol{p}) = \boldsymbol{\beta}_0^*(\boldsymbol{p}) \qquad \boldsymbol{\vartheta}(\boldsymbol{p}) = \boldsymbol{\vartheta}_\ell(\boldsymbol{p}) = \boldsymbol{\vartheta}_0^*(\boldsymbol{p}) \tag{6.9}$$

in other words, the Galilean and Lorentz sections are duals to each other.

(3) The symmetric section σ_s

This section is self-dual, being given by

$$\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_s(\boldsymbol{p}) = \boldsymbol{\beta}_s^*(\boldsymbol{p}) = \frac{\boldsymbol{p}}{m + p_0} \qquad \boldsymbol{\vartheta}(\boldsymbol{p}) = \boldsymbol{\vartheta}_s(\boldsymbol{p}) = \boldsymbol{\vartheta}_s^*(\boldsymbol{p}) = \frac{\boldsymbol{p}}{m + p_0}. \tag{6.10}$$

Again, $\|\boldsymbol{\beta}_s(\boldsymbol{p})\| < 1,\ \forall\ \boldsymbol{p}$. Note that in a sense σ_s *lies half-way* between σ_0 and σ_ℓ. Indeed, writing

$$n_g = (1, \mathbf{0}) \qquad n_s = \left(\left[\frac{p_0 + m}{2m}\right]^{\frac{1}{2}}, \left[\frac{1}{2m(p_0 + m)}\right]^{\frac{1}{2}} \boldsymbol{p}\right) \qquad n_\ell = \frac{1}{m}(p_0, \boldsymbol{p}) \tag{6.11}$$

we find that

$$\begin{aligned} &n_\ell = n_g^* = \Lambda_p \overline{n_g} = \Lambda_p n_g \qquad n_s^* = n_s \\ &\Lambda_{(mn_s)}^2 = \Lambda_p \qquad n_\ell = \Lambda_{(mn_s)} n_s = \Lambda_{(mn_s)}^2 n_g \end{aligned} \tag{6.12}$$

so that the velocity $\boldsymbol{\beta}_s$ lies half-way between $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_\ell$.

(4) The limiting sections $\sigma_\pm$

These sections are duals of each other and are both light-like, being given by

$$\begin{aligned} &\boldsymbol{\beta}(\boldsymbol{p}) = \boldsymbol{\beta}_+(\boldsymbol{p}) = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} \qquad \boldsymbol{\vartheta}(\boldsymbol{p}) = \boldsymbol{\vartheta}_+(\boldsymbol{p}) = \frac{m\boldsymbol{p}}{\|\boldsymbol{p}\|(p_0 - \|\boldsymbol{p}\|)} \\ &\boldsymbol{\beta}_-(\boldsymbol{p}) = \boldsymbol{\beta}_+^*(\boldsymbol{p}) = -\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} \qquad \boldsymbol{\vartheta}_-(\boldsymbol{p}) = \boldsymbol{\vartheta}_+^*(\boldsymbol{p}) = -\frac{m\boldsymbol{p}}{\|\boldsymbol{p}\|(p_0 + \|\boldsymbol{p}\|)}. \end{aligned} \tag{6.13}$$

In this limiting situation, $\|\boldsymbol{\beta}_+(\boldsymbol{p})\| = \|\boldsymbol{\beta}_-(\boldsymbol{p})\| = 1,\ \forall\ \boldsymbol{p}$.

We end this discussion by analysing the relationship between physical events observed from reference frames attached to different sections. We identify the rest or laboratory frame with the Galilean frame K_g, i.e. we associate the laboratory with the Galilean section σ_0. Similarly, to each $\boldsymbol{\beta}$ we attach a frame K_β (understood pointwise, for each $\boldsymbol{p}$), moving with velocity $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{p})$ with respect to K_g. In other words, K_β is the frame attached to the space-like affine section σ_β. In particular, to the Lorentz section σ_ℓ we attach the reference frame K_ℓ, moving with velocity $\boldsymbol{p}/p_0$ with respect to the Galilean frame. As before, let $u(p) = m\{n(p)\}$, with $n(p)$ the 4-velocity corresponding to $\boldsymbol{\beta}(\boldsymbol{p})$ (see equation (5.13)). Then, following (2.9), the Lorentz transformation which boosts K_g to K_β is

$$\Lambda_\beta = \frac{1}{m}\begin{pmatrix} u_0 = (1 - \|\boldsymbol{\beta}\|^2)^{\frac{1}{2}} & \boldsymbol{u}^\dagger \\ \boldsymbol{u} & mV_u \end{pmatrix}. \tag{6.14}$$

Let $(q_0^\beta,\ \boldsymbol{q}^\beta)$ be the coordinates of an event when seen from K_β. The corresponding coordinates with respect to K_g (see equation (2.14)) are

$$\begin{aligned}\Lambda_\beta(q_0^\beta,\ \boldsymbol{q}^\beta) &= \left(\frac{1}{m}(u_0 q_0 + \boldsymbol{u}\cdot\boldsymbol{q}^\beta\right),\ \frac{1}{m}\left(\boldsymbol{u}q_0 + mV_u\boldsymbol{q}^\beta\right) \\ &= (q_0^g,\ \boldsymbol{q}^g)\end{aligned} \tag{6.15}$$

while those with respect to K_ℓ are

$$(q_0^\ell,\ \boldsymbol{q}^\ell) = \Lambda_{\overline{p}}\Lambda_\beta(q_0^\beta,\ \boldsymbol{q}^\beta). \tag{6.16}$$

Here

$$q_0^\ell = \frac{1}{m^2}(u_0 p_0 - \boldsymbol{u}\cdot\boldsymbol{p})q_0^\beta + \frac{1}{m^2}u_0 p_0 \boldsymbol{u}\cdot\boldsymbol{q}^\beta - \frac{1}{m}\boldsymbol{p}\cdot V_u\boldsymbol{q}^\beta, \tag{6.17}$$

$$\boldsymbol{q}^\ell = -\frac{1}{m^2}(u_0\boldsymbol{p} - mV_p\boldsymbol{u})q_0^\beta - \frac{1}{m^2}u_0\boldsymbol{p}\boldsymbol{u}\cdot\boldsymbol{q}^\beta + V_pV_u\boldsymbol{q}^\beta. \tag{6.18}$$

Consider all events lying in Σ_p^β, i.e. all events for which $q_0^\beta = 0$. Combining equation (6.15) with (6.17) and (6.18) and applying (6.6) to get the inverse relation, we obtain

$$\boldsymbol{q}^\ell = \left(V_p - \frac{\boldsymbol{p}\otimes\boldsymbol{\beta}^\dagger}{m}\right)\boldsymbol{q}^g \qquad \boldsymbol{q}^g = \left(V_p - \frac{\boldsymbol{p}\otimes\boldsymbol{\beta}^{*\dagger}}{m}\right)\boldsymbol{q}^\ell. \tag{6.19}$$

In particular, if $\boldsymbol{\beta} = \boldsymbol{\beta}_s$, the symmetric section, then $\boldsymbol{q}^\ell$ and $\boldsymbol{q}^g$ coincide. Thus the spatial coordinates of an event localized to a space-like hyperplane Σ_p^s of the symmetric section, appear same when observed from the laboratory or the Lorentz frame. Similarly, if we consider events for which $\boldsymbol{q}^\beta = 0$, then as before we obtain (see equation (6.5))

$$q_0^\ell = \frac{1}{m}(p_0 - \boldsymbol{p}\cdot\boldsymbol{\beta})q_0^g \qquad q_0^g = \frac{1}{m}(p_0 - \boldsymbol{p}\cdot\boldsymbol{\beta}^*)q_0^\ell. \tag{6.20}$$

Once again, considering the symmetric section, the temporal coordinates of an event taking place at the spatial origin of a space-like hyperplane Σ_p^s, appear same when observed from the laboratory or the Lorentz frame. These two sets of relations point up yet another aspect of the $\boldsymbol{\beta} - \boldsymbol{\beta}^*$-duality.

Finally, it is tempting to speculate on the meaning of certain pseudo-differential equations arising from the quantized versions of the two relations (6.5) and (6.6). Making the replacement $p_j \to -\mathrm{i}\partial/\partial q_j,\ j = 1, 2, 3$, turns $p_0 - \boldsymbol{p}\cdot\boldsymbol{\beta}(\boldsymbol{p})$ into a pseudo-differential operator and $mV_p - \boldsymbol{p}\otimes\boldsymbol{\beta}(\boldsymbol{p})^\dagger$ into a matrix pseudo-differential operator. Let

$$D = \begin{pmatrix} p_0 - \boldsymbol{p}\cdot\boldsymbol{\beta}(\boldsymbol{p}) & \mathbf{0}^\dagger \\ \mathbf{0} & mV_p - \boldsymbol{p}\otimes\boldsymbol{\beta}(\boldsymbol{p})^\dagger \end{pmatrix} \tag{6.21}$$

$$D_* = \begin{pmatrix} p_0 - \boldsymbol{p}\cdot\boldsymbol{\beta}^*(\boldsymbol{p}) & \mathbf{0}^\dagger \\ \mathbf{0} & mV_p - \boldsymbol{p}\otimes\boldsymbol{\beta}^*(\boldsymbol{p})^\dagger \end{pmatrix} \tag{6.22}$$

both considered as matrix pseudo-differential operators. If ψ_0 and ψ_1 are each 4-component wavefunctions, relations (6.5) and (6.6) together give rise to the 8×8 matrix pseudo-differential equation

$$\frac{1}{m}\begin{pmatrix} 0 & D \\ D_* & 0 \end{pmatrix}\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \tag{6.23}$$

or equivalently

$$DD_*\psi_0 = m^2\psi_0 \qquad \text{and} \qquad D_*D\psi_1 = m^2\psi_1. \tag{6.24}$$

It would be interesting to investigate, both physically and mathematically, the meaning of these Dirac-type equations.

7. Spin-*s* frames and coherent states

From now on, unless otherwise stated, we shall work with space-like affine sections σ; actually the only exceptions will be the two limiting sections $\sigma_\pm$ in (6.13). Thus we shall assume that the conditions of proposition 4.1 hold.

Going back to the computation of $I_{\phi,\psi}$ in (3.21), we note that the d$\boldsymbol{q}$ integration yields a δ-measure in $\boldsymbol{X}$, and hence, making the change of variables $\boldsymbol{k} \to \boldsymbol{X}$, integrating and rearranging (using equation (4.1)) we obtain

$$I_{\phi,\psi} = \int_{\mathcal{V}_m^+\times\mathcal{V}_m^+} \boldsymbol{\phi}(k)^\dagger \mathcal{A}_\sigma(k,p)\boldsymbol{\psi}(k)\ \frac{\mathrm{d}\boldsymbol{p}}{p_0}\frac{\mathrm{d}\boldsymbol{k}}{k_0} \tag{7.1}$$

where $\mathcal{A}_\sigma(k,p)$ is the $(2s+1)\times(2s+1)$-matrix kernel

$$\begin{aligned}\mathcal{A}_\sigma(k,p) = (2\pi)^3 m \sum_{i=1}^{2s+1}&\left[p_0 + \boldsymbol{p}\cdot\rho(k\to\Lambda_k^{-1}p)^\dagger\ \boldsymbol{\beta}^*\left(-\underline{\Lambda_k^{-1}p}\right)\right]^{-1}\\ &\times\mathcal{D}^s(v(k,\overline{\Lambda_k^{-1}p}))\ \boldsymbol{\eta}^i(\rho(\overline{\Lambda_k^{-1}p})^{-1}\rho(k\to\Lambda_k^{-1}p)p)\\ &\times\boldsymbol{\eta}^i(\rho(\overline{\Lambda_k^{-1}p})^{-1}\rho(k\to\Lambda_k^{-1}p)p)^\dagger\ \mathcal{D}^s(v(k,\overline{\Lambda_k^{-1}p}))^\dagger\end{aligned} \tag{7.2}$$

where $\rho(p)$, $\rho(k\to p)$ and $v(k,p)$ are given by equations (3.17), (4.2) and (3.20), respectively. Assuming the integral (3.16) to exist for all $\boldsymbol{\phi},\boldsymbol{\psi}\in\mathfrak{H}_W^s$, let us write

$$\mathcal{A}_\sigma(k) = \int_{\mathcal{V}_m^+} \mathcal{A}_\sigma(k,p)\ \frac{\mathrm{d}\boldsymbol{p}}{p_0}. \tag{7.3}$$

Then the operator A_σ in (3.14) is a matrix-valued multiplication operator:

$$(A_\sigma\boldsymbol{\phi})(k) = \mathcal{A}_\sigma(k)\boldsymbol{\phi}(k) \qquad \boldsymbol{\phi}\in\mathfrak{H}_W^s. \tag{7.4}$$

At this point we make two simplifying assumptions on the nature of the vectors $\boldsymbol{\eta}^i\in\mathfrak{H}_W^s$, $i = 1,2,\ldots,2s+1$.

(i) Assumption of rotational invariance of the operator $\sum_{i=1}^{2s+1}|\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|$, i.e. $\forall\ R\in SU(2)$

$$\mathcal{D}^s(R)\left[\sum_{i=1}^{2s+1}|\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|\right]\mathcal{D}^s(R)^\dagger = \sum_{i=1}^{2s+1}|\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|. \tag{7.5}$$

This implies that

$$\left(\sum_{i=1}^{2s+1}|\boldsymbol{\eta}^i\rangle\langle\boldsymbol{\eta}^i|\right)(k) = \mathbb{I}_{2s+1}\ |\eta(k)|^2 \tag{7.6}$$

where $\eta \in L^2(\mathcal{V}_m^+, \mathrm{d}\boldsymbol{k}/k_0)$, and thus we may simply take for $\boldsymbol{\eta}^i \in \mathfrak{H}_W^s$ the vectors

$$\boldsymbol{\eta}^i = \hat{\boldsymbol{e}}_i \otimes \eta \qquad i = 1, 2, \ldots, 2s+1 \tag{7.7}$$

the $\hat{\boldsymbol{e}}_i$ being the canonical unit vectors in $\mathbb{C}^{2s+1}$ (i.e. $\hat{\boldsymbol{e}}_i = (\delta_{ij}),\ j = 1, 2, \ldots, 2s+1$).
(ii) Assumption of rotational invariance of $|\eta(k)|^2$ in (7.6):

$$|\eta(\rho k)|^2 = |\eta(k)|^2 \qquad \forall\ \rho \in SO(3). \tag{7.8}$$

We shall generally refer to these two assumptions as the *assumption of rotational invariance*. With this, the kernel $\mathcal{A}_\sigma(k, p)$ in (7.2) simplifies to

$$\begin{aligned} &\mathcal{A}_\sigma(k, p) = a_\sigma(k, p)\ |\eta(p)|^2\ \mathbb{I}_{2s+1} \\ &a_\sigma(k, p) = \frac{(2\pi)^3 m}{p_0 + \boldsymbol{p}\cdot\rho(k \to \Lambda_k^{-1} p)^\dagger\ \boldsymbol{\beta}^*\left(-\underline{\Lambda_k^{-1} p}\right)}. \end{aligned} \tag{7.9}$$

On $\mathfrak{H}_W^s$ define the operators $(P_0, \boldsymbol{P})$

$$(P_\mu \phi)(k) = k_\mu \phi(k). \tag{7.10}$$

We shall also denote the analogous operators on $L^2(\mathcal{V}_m^+, \mathrm{d}\boldsymbol{k}/k_0)$ by the same symbols. P_0^{-1} is a bounded operator with spectrum $[0, \frac{1}{m}]$. With the above simplifications (7.3) becomes

$$\mathcal{A}_\sigma(k) = \langle a_\sigma(k, P)\rangle_\eta\ \mathbb{I}_{2s+1} \tag{7.11}$$

where $\langle\cdot\rangle_\eta$ denotes the $L^2(\mathcal{V}_m^+, \mathrm{d}\boldsymbol{k}/k_0)$ expectation value with respect to the vector η in (7.6). Hence for the operator A_σ (see equation (7.4))

$$\|A_\sigma\| = \sup_{k \in \mathcal{V}_m^+} |\langle a_\sigma(k, P)\rangle_\eta| \tag{7.12}$$

provided this supremum exists. On the other hand, since $\|\boldsymbol{\beta}^*\left(-\underline{\Lambda_k^{-1} p}\right)\| < 1$ and $\|\rho(k \to \Lambda_k^{-1} p)^\dagger\| = 1$, from (7.9) we get

$$\frac{1}{(2\pi)^3 m}(p_0 - \|\boldsymbol{p}\|) < \frac{1}{a_\sigma(k, p)} < \frac{1}{(2\pi)^3 m}(p_0 + \|\boldsymbol{p}\|)\,. \tag{7.13}$$

The two extreme values in the above inequality are actually reached for the limiting sections $\sigma_\pm$ (see (6.10)). Thus, we have the following result.

Lemma 7.1. If $\|\boldsymbol{\beta}(\boldsymbol{p})\| \leqslant 1,\ \ \forall\ \boldsymbol{p}$, then $a_\sigma(k, p)$ is a bounded function satisfying

$$\frac{(2\pi)^3}{m}(p_0 - \|\boldsymbol{p}\|) \leqslant a_\sigma(k, p) \leqslant \frac{(2\pi)^3}{m}(p_0 + \|\boldsymbol{p}\|). \tag{7.14}$$

Suppose now that η lies in the domain of $P_0^{\frac{1}{2}}$, i.e.

$$\int_{\mathcal{V}_m^+} |\eta(k)|^2\ \mathrm{d}\boldsymbol{k} < \infty \tag{7.15}$$

and set

$$\langle P_0 \pm \|\boldsymbol{P}\|\rangle_\eta = \int_{\mathcal{V}_m^+} (p_0 \pm \|\boldsymbol{p}\|)|\eta(p)|^2\ \frac{\mathrm{d}\boldsymbol{p}}{p_0}. \tag{7.16}$$

Then equations (7.1), (7.9) and (7.14) together imply the following.

Lemma 7.2. If the assumption of rotational invariance on $\boldsymbol{\eta}^i, \quad i = 1, 2, \ldots, 2s+1$ is satisfied, and if $\eta \in \mathrm{Dom}(P_0^{\frac{1}{2}})$, then for all $\boldsymbol{\beta}$ such that $\|\boldsymbol{\beta}(\boldsymbol{p})\| \leqslant 1, \quad \forall\, \boldsymbol{p}$

$$\frac{(2\pi)^3}{m}\langle P_0 - \|\boldsymbol{P}\|\rangle_\eta \, \|\boldsymbol{\phi}\| \, \|\boldsymbol{\psi}\| \leqslant |I_{\phi,\psi}| \leqslant \frac{(2\pi)^3}{m}\langle P_0 + \|\boldsymbol{P}\|\rangle_\eta \, \|\boldsymbol{\phi}\| \, \|\boldsymbol{\psi}\|. \tag{7.17}$$

As a consequence of this lemma we see that both the operator A_σ in (3.14) and its inverse, A_σ^{-1}, are bounded, with

$$(A_\sigma^{-1}\boldsymbol{\phi})(k) = [\langle a_\sigma(k, P)\rangle_\eta]^{-1}\boldsymbol{\phi}(k) \qquad \boldsymbol{\phi} \in \mathfrak{H}_W^s. \tag{7.18}$$

Indeed, collecting all these results we obtain the following.

Proposition 7.3. Let $\boldsymbol{\eta}^i, \quad i = 1, 2, \ldots, 2s+1$, satisfy the condition of rotational invariance. Then for each $\boldsymbol{\beta}$ satisfying $\|\boldsymbol{\beta}(\boldsymbol{p})\| \leqslant 1, \quad \forall\, \boldsymbol{p}$, the set of vectors $\mathfrak{S}_\sigma$ in (3.15) is a family of spin-s coherent states, forming a rank-$(2s+1)$ frame $\mathcal{F}\{\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})}, A_\sigma, 2s+1\}$, if and only if $\eta \in \mathrm{Dom}\,(P_0^{\frac{1}{2}})$. The operator A_σ acts via multiplication by a bounded invertible function $\mathcal{A}_\sigma(k)$ given by (7.11) and A_σ^{-1} via multiplication by the function $\mathcal{A}_\sigma^{-1}(k)$. Moreover

$$\frac{(2\pi)^3}{m}\langle P_0 - \|\boldsymbol{P}\|\rangle_\eta \leqslant \|A_\sigma\| \leqslant \frac{(2\pi)^3}{m}\langle P_0 + \|\boldsymbol{P}\|\rangle_\eta \tag{7.19}$$

and

$$\mathrm{Spectrum}(A_\sigma) \subset \frac{(2\pi)^3}{m}\left[\langle P_0 - \|\boldsymbol{P}\|\rangle_\eta, \langle P_0 + \|\boldsymbol{P}\|\rangle_\eta\right]. \tag{7.20}$$

Note that since we are assuming rotational invariance, we could just as well have done without the restriction, $R(\boldsymbol{q}, \boldsymbol{p}) = R(\boldsymbol{p})$, in defining the sections σ in (3.9). The following construction now emerges for building spin-s coherent states for the representations U_W^s (see (2.6)) of mass $m > 0$ and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, of $\mathcal{P}_+^\uparrow(1, 3)$:

(i) Choose a function $\boldsymbol{\beta} : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\|\boldsymbol{\beta}(\boldsymbol{p})\| \leqslant 1, \quad \forall\, \boldsymbol{p}$, or equivalently, a map $u : \mathcal{V}_m^+ \to \mathcal{V}_m^+$ as in (6.1); choose an arbitrary measurable function $R : \mathbb{R}^3 \to SU(2)$ and construct the corresponding affine space-like (or in the limit, the affine light-like) section σ, using equations (3.9)–(3.11).
(ii) Choose an $\eta \in L^2(\mathcal{V}_m^+, \mathrm{d}\boldsymbol{k}/k_0)$, satisfying equations (7.8) and (7.15) and form the vectors $\boldsymbol{\eta}^i, \quad i = 1, 2, \ldots, 2s+1$, using equation (7.7).
(iii) Construct the family, $\mathfrak{S}_\sigma$, of coherent states $\boldsymbol{\eta}^i_{\boldsymbol{q},\boldsymbol{p})}$ using equation (3.14).

While this procedure provides us with a large class of CS and frames, the latter are generally not tight, i.e. A_σ is not, in general, a multiple of the identity. A few special cases worked out below will make this statement clearer. For computational purposes, the following expressions prove useful (assuming rotational invariance):

$$a_\sigma(k, p) = \frac{(2\pi)^3 m(\Lambda_k^{-1}p)_0}{mk_0 - \left[k_0\left(\underline{\Lambda_k^{-1}p}\right) + \boldsymbol{k}(\Lambda_k^{-1}p)_0\right]\cdot\boldsymbol{\vartheta}\left(-\underline{\Lambda_k^{-1}p}\right)} \tag{7.21}$$

and

$$I_{\phi,\psi} = (2\pi)^3 \int_{\mathcal{V}_m^+ \times \mathcal{V}_m^+} \boldsymbol{\phi}(k)^\dagger \frac{m}{mk_0 - (k_0\boldsymbol{p} + \boldsymbol{k}p_0)\cdot\boldsymbol{\vartheta}(-\boldsymbol{p})} |\eta(\Lambda_k p)|^2 \boldsymbol{\psi}(k) \,\mathrm{d}\boldsymbol{p}\, \frac{\mathrm{d}\boldsymbol{k}}{k_0}. \tag{7.22}$$

(1) The Galilean section σ_0

From equations (6.8) and (7.21)

$$a_\sigma(k, p) = a_0(k, p) = \frac{(2\pi)^3}{m}\ \frac{k_0 p_0 - \boldsymbol{k}\cdot\boldsymbol{p}}{k_0} \tag{7.23}$$

and using the rotational invariance of $|\eta(k)|^2$

$$\mathcal{A}_\sigma(k) = \mathcal{A}_0(k) = \frac{(2\pi)^3}{m}\langle P_0\rangle_\eta\ \mathbb{I}_{2s+1}. \tag{7.24}$$

Hence

$$A_\sigma = A_0 = \frac{(2\pi)^3}{m}\langle P_0\rangle_\eta\ I \tag{7.25}$$

so that the frame is tight.

(2) The Lorentz section σ_ℓ

From equations (6.9) and (7.9)

$$a_\sigma(k, p) = a_\ell(k, p) = \frac{(2\pi)^3 m}{p_0} \tag{7.26}$$

so that

$$\mathcal{A}_\sigma(k) = \mathcal{A}_\ell(k) = (2\pi)^3 m\langle P_0^{-1}\rangle_\eta\ \mathbb{I}_{2s+1}. \tag{7.27}$$

Thus

$$A_\sigma = A_\ell = (2\pi)^3 m\langle P_0^{-1}\rangle_\eta\ I \tag{7.28}$$

and once again the frame is tight.

(3) The symmetric section σ_s

From equations (6.10) and (7.22)

$$I_{\phi,\psi} = (2\pi)^3\int_{\mathcal{V}_m^+\times\mathcal{V}_m^+}\phi(k)^\dagger\ \frac{k_0 p_0 + m^2}{m(k_0+p_0)}|\eta(p)|^2\psi(k)\ \frac{\mathrm{d}\boldsymbol{p}}{p_0}\frac{\mathrm{d}\boldsymbol{k}}{k_0}. \tag{7.29}$$

Thus

$$a_\sigma(k, p) = a_s(k, p) = (2\pi)^3\ \frac{k_0 p_0 + m^2}{m(k_0+p_0)} \tag{7.30}$$

and

$$\mathcal{A}_\sigma(k) = \mathcal{A}_s(k) = (2\pi)^3\left\langle\frac{k_0 P_0 + m^2}{m(k_0+P_0)}\right\rangle_\eta\ \mathbb{I}_{2s+1}. \tag{7.31}$$

The operator $A_\sigma = A_s$ is now given by

$$(A_s\phi)(k) = \mathcal{A}_s(k)\phi(k) = (2\pi)^3\left\langle\frac{k_0 P_0 + m^2}{m(k_0+P_0)}\right\rangle_\eta\ \phi(k) \qquad \phi\in\mathfrak{H}_W^s. \tag{7.32}$$

To determine the spectrum of A_s, note that the function $f : [m,\infty)\to\mathbb{R}^+$, defined by

$$f(k_0) = (2\pi)^3\frac{k_0 p_0 + m^2}{m(k_0+p_0)} \tag{7.33}$$

is uniformly bounded for all $p_0 \in [m, c]$, for any finite $c > m$. Also, $f'(k_0) \neq 0$, for all $k_0, p_0 > m$ and, $f(m) = (2\pi)^3$, $f(\infty) = (2\pi)^3 p_0/m$. Thus, $(2\pi)^3 \leqslant f(k_0) \leqslant (2\pi)^3 p_0/m$, which, by virtue of (7.31), implies that

$$\text{Spectrum}(A_s) = (2\pi)^3 \left[\|\eta\|^2, \ \frac{\langle P_0 \rangle_\eta}{m} \right]. \tag{7.34}$$

Hence, in this case, the frame is never tight.

We end by proving the stability of the class of affine sections under the action of $\mathcal{P}_+^\uparrow(1,3)$. If $\sigma : \Gamma \to \mathcal{P}_+^\uparrow(1,3)$ is any section then, as shown in [4], for arbitrary $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$, $\sigma_{(a,A)}$ is again a section where

$$\sigma_{(a,A)}(\boldsymbol{q}, \boldsymbol{p}) = (a, A)\sigma((a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})) = \sigma(\boldsymbol{q}, \boldsymbol{p})\, h((a, A),\ (a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})) \tag{7.35}$$

$(a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})$ being the translation of $(\boldsymbol{q}, \boldsymbol{p})$ by $(a, A)^{-1}$ under the action (3.5) and

$$h((a, A),\ (a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})) = \sigma(\boldsymbol{q}, \boldsymbol{p})^{-1}(a, A)\sigma((a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})) \in T \times SU(2)\,.$$

Moreover, if σ defines the frame $\mathcal{F}\{\boldsymbol{\eta}^i_{\sigma(\boldsymbol{q},\boldsymbol{p})}, A_\sigma, 2s+1\}$, then $\sigma_{(a,A)}$ defines the frame $\mathcal{F}\{\boldsymbol{\eta}^i_{\sigma_{(a,A)}(\boldsymbol{q},\boldsymbol{p})}, A_{\sigma_{(a,A)}}, 2s+1\}$, where $A_{\sigma_{(a,A)}} = U(a, A)A_\sigma U(a, A)^*$. Let $\mathfrak{A}$ denote the class of all affine space-like sections, defined by (3.8) – (3.10) and satisfying the conditions of proposition 4.1, but with φ not necessarily assumed to be zero. Then:

Proposition 7.4. If $\sigma \in \mathfrak{A}$ then $\sigma_{(a,A)} \in \mathfrak{A}$, for all $(a, A) \in \mathcal{P}_+^\uparrow(1,3)$.

In view of this result (the proof is given in the appendix), starting with any family of coherent states $\mathfrak{S}_\sigma$, we may generate an entire class of covariantly translated families $\mathfrak{S}_{\sigma_{(a,A)}}$ of other coherent states, using the natural action (7.35) of $\mathcal{P}_+^\uparrow(1,3)$ on the space of sections. If σ is characterized by $\boldsymbol{\beta}$ and φ, then $\sigma_{(a,A)}$ is characterized by $\boldsymbol{\beta}'$ and φ', the relationship between them being given by (A.7) below.

8. Some specific applications

Coherent states enable us to view states and observables of quantum mechanical systems in a very special manner. For instance, if we consider the hydrogen atom, we can transcribe its continuum and bound state wavefunctions, dipole operators, etc., in terms of an overcomplete basis consisting of Galilean CS built up as:

$$\langle \boldsymbol{r} | \boldsymbol{q}, \boldsymbol{p} \rangle = \left(U(\sigma_0(\boldsymbol{q}, \boldsymbol{p})) \sum_{n,\ell,m} S_{n,\ell,m} \right) (p_0, \boldsymbol{r})$$

where the $S_{n,\ell,m}(p_0, \boldsymbol{r})$ are the so-called Sturmian functions [15]. The Sturmians form a complete discrete basis set in the Hilbert space. The states $\langle \boldsymbol{r} | \boldsymbol{q}, \boldsymbol{p} \rangle$ are especially adapted to computations of, for example, matrix elements of multi-photon processes in the Galilean regime [12], [17]. The same holds true in the intermediate relativistic regime, where the Dirac or the Feynman–Gell–Mann equation, with external field, remains valid for describing the interaction of a charged spin-$\frac{1}{2}$ particle with the electromagnetic field (and where a full QED model is not warranted). For recent work in atomic physics in this direction, see, e.g., [18]. The spin-Sturmian functions for the Feynman–Gell–Mann equation were obtained in [8]. These wavefunctions can be used to build relativistic CS in the spirit of the present paper:

$$\langle \boldsymbol{r} | \boldsymbol{q}, \boldsymbol{p}, s \rangle = \left(U^s_W(\sigma(\boldsymbol{q}, \boldsymbol{p})) \sum_{n,\ell,m} S^s_{n,\ell,m} \right) (p_0, \boldsymbol{r})\,.$$

The freedom in the choice of available sections, in building the coherent states, can now be exploited, in addition to the freedom which already exists in the choice of the Sturmian probe. The various sections σ, discussed in this paper, also have applications to relativistic statistical mechanics in the computation of distribution functions [11].

Appendix

Derivation of equation (4.1)

Rewriting (4.2) as $\rho(k \to \overline{p})\Lambda_k \Lambda_p^{-1} = \Lambda_p^{-1}\Lambda_k$, and acting on the vector $(m, \mathbf{0})$ with both sides of this equation we obtain

$$\rho(k \to \overline{p})\Lambda_k \overline{p} = \Lambda_p^{-1} k\,. \tag{A.1}$$

Next, by equation (3.13),

$$k_0 + \frac{1}{m}\,(k_0\boldsymbol{p} - \boldsymbol{k}p_0)\cdot\boldsymbol{\vartheta}(\boldsymbol{p}) = k_0 + \frac{1}{m}\,(k_0\boldsymbol{p} - \boldsymbol{k}p_0)\cdot\left(\frac{\boldsymbol{p}}{m} - V_p\boldsymbol{\beta}^*(\boldsymbol{p})\right).$$

Using the explicit form of V_p from (2.10) and simplifying we get

$$\begin{aligned} k_0 + \frac{1}{m}\,(k_0\boldsymbol{p} - \boldsymbol{k}p_0)\cdot\boldsymbol{\vartheta}(\boldsymbol{p}) &= \frac{p_0}{m^2} k\cdot p - \frac{p_0}{m}\left[\frac{k_0\boldsymbol{p}}{m} - V_p\boldsymbol{k}\right]\cdot\boldsymbol{\beta}^*(\boldsymbol{p}) \\ &= \frac{p_0}{m}\left[(\Lambda_k p)_0 + \left(\underline{\Lambda_{\overline{p}} k}\right)\cdot\boldsymbol{\beta}^*(\boldsymbol{p})\right] \end{aligned}$$

by virtue of (2.14). From this equation (4.1) follows directly upon using (A.1).

Proof of proposition 4.1

We start with (i). The condition rewritten as $\|\hat{\boldsymbol{q}}\|^2 - |\hat{q}_0|^2 > 0$, implies by (3.10) and (3.11) that $\boldsymbol{q}\cdot S(\boldsymbol{p},\boldsymbol{\vartheta})\boldsymbol{q} > 0,\ \forall\ \boldsymbol{q}\in\mathbb{R}^3,\ \boldsymbol{q}\neq\mathbf{0}$, which is equivalent to (ii). The equivalence of (i) and (ii) follows directly from (3.10), while the equivalence of (iii) and (iv) has been established in (6.7). □

Proof of proposition 4.2

Suppose that $\hat{q}$ is space-like. Then by proposition 4.1, $\|\boldsymbol{\beta}^*\| < 1$. Hence, since $\rho(k \to \hat{p})$ in (4.1) is a rotation matrix

$$\left\|\frac{(\Lambda_k\overline{p})}{(\Lambda_k\overline{p})_0}\cdot\rho(k\to\overline{p})^\dagger\boldsymbol{\beta}^*(\boldsymbol{p})\right\| < 1 \quad\Rightarrow\quad 1 + \frac{(\Lambda_k\overline{p})}{(\Lambda_k\overline{p})_0}\cdot\rho(k\to\overline{p})^\dagger\boldsymbol{\beta}^*(\boldsymbol{p}) > 0$$

i.e. $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] > 0$ (by equation (4.1)).

Conversely, assume that $\det[\mathcal{J}_{\boldsymbol{X}}(\boldsymbol{k})] > 0$. Then by equation (3.23)

$$1 + \frac{\boldsymbol{p}\cdot\boldsymbol{\vartheta}(\boldsymbol{p})}{m} > \frac{p_0}{m}\,\frac{\boldsymbol{k}}{k_0}\cdot\boldsymbol{\vartheta}(\boldsymbol{p})\,.$$

Taking $\boldsymbol{k}$ in the direction of $\boldsymbol{\vartheta}(\boldsymbol{p})$ and letting $\|\boldsymbol{k}\| \to \infty$, the above inequality implies that

$$1 + \frac{\boldsymbol{p}\cdot\boldsymbol{\vartheta}(\boldsymbol{p})}{m} > \frac{p_0}{m}\,\|\boldsymbol{\vartheta}(\boldsymbol{p})\|$$

which, in view of condition (iii) of proposition 4.1, implies that $\hat{q}$ is space-like. □

Proof of proposition 7.4

If φ is included in the definition of the section, it is easily checked that

$$\hat{q}_0 = \frac{p_0}{m}\ (\varphi(\boldsymbol{p}) + \boldsymbol{\vartheta}(\boldsymbol{p}) \cdot \boldsymbol{q}) \qquad \hat{\boldsymbol{q}} = \frac{\boldsymbol{p}}{m}\varphi(\boldsymbol{p}) + \boldsymbol{q} + \frac{1}{m}\boldsymbol{p}\boldsymbol{\vartheta}(\boldsymbol{p}) \cdot \boldsymbol{q}. \tag{A.2}$$

From this it follows that

$$\hat{q}_0 = \frac{p_0}{m}\varphi(\boldsymbol{p}) + \boldsymbol{\beta}(\boldsymbol{p}) \cdot (\hat{\boldsymbol{q}} - \frac{\boldsymbol{p}}{m}\varphi(\boldsymbol{p})) \quad \Rightarrow \quad n(p) \cdot \hat{q} = \frac{n(p) \cdot p}{m}\varphi(\boldsymbol{p}) \tag{A.3}$$

with $n(p)$ given by (5.13). Next, write $\Lambda = \Lambda_k \rho$, where ρ is a rotation. Then $\Lambda^{-1} = \Lambda_{\rho^{-1}\underline{k}}\ \rho^{-1}$, so that writing $(\boldsymbol{q}', \boldsymbol{p}') = (a, A)^{-1}(\boldsymbol{q}, \boldsymbol{p})$ in (7.35), we get (see equation (3.5))

$$\boldsymbol{q}' = -\frac{a_0}{m}\rho^{-1}\boldsymbol{k} + \rho^{-1}V_k(\boldsymbol{q} - \boldsymbol{a}) \qquad \boldsymbol{p}' = \underline{\Lambda^{-1}p} \tag{A.4}$$

with V_k as in (2.10).

Thus, if σ is the affine section corresponding to the quantities $\boldsymbol{\beta}$ and φ, and $(\hat{q}', p') = \sigma(\boldsymbol{q}', \boldsymbol{p}')$, then

$$n(p') \cdot \hat{q}' = \frac{n(p') \cdot p'}{m}\varphi(\boldsymbol{p}'). \tag{A.5}$$

Let

$$(\hat{q}'', h(p)) = (a, A)\sigma(\boldsymbol{q}', \boldsymbol{p}') = (a, A)(\hat{q}', h(p')) = (a + \Lambda\hat{q}', h(p))$$

and $n'(p) = \Lambda n(p)$. Clearly, $\boldsymbol{\beta}' = \underline{\Lambda n}/(\Lambda n)_0$ satisfies $\|\boldsymbol{\beta}'\| \leqslant 1$ if $\|\boldsymbol{\beta}\| \leqslant 1$. Furthermore

$$\begin{aligned} n'(p) \cdot \hat{q}'' &= n'(p) \cdot (a + \Lambda\hat{q}') \\ &= \frac{n'(p) \cdot p}{m}\left[\frac{n(p) \cdot (m\Lambda^{-1}a + \varphi(\boldsymbol{p})p)}{n'(p) \cdot p}\right]. \end{aligned} \tag{A.6}$$

Thus, $\sigma_{(a,A)}(\boldsymbol{q}, \boldsymbol{p})$ is again an affine section corresponding to the quantities $\boldsymbol{\beta}'$ and φ', with

$$\boldsymbol{\beta}'(\boldsymbol{p}) = \frac{\underline{\Lambda n(p)}}{(\Lambda n(p))_0} \qquad \varphi'(\boldsymbol{p}) = \frac{n(p) \cdot (m\Lambda^{-1}a + \varphi(\boldsymbol{p})p)}{n'(p) \cdot p}. \tag{A.7}$$

□

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